

# EXPLICIT COPRODUCT FORMULA FOR QUANTUM GROUP OF THE TYPE $G_2$

V.K. KHARCHENKO, C. VAY

**ABSTRACT.** We find a coproduct formula in the explicit form for PBW-generators of the two-parameter quantum group  $U_q^+(\mathfrak{g})$  where  $\mathfrak{g}$  is a simple Lie algebra of type  $G_2$ . The similar formulas for quantizations of simple Lie algebras of infinite series are already known.

## 1. INTRODUCTION

In the present paper, we establish an explicit coproduct formula for the quantum group  $U_q(\mathfrak{g})$ , where  $\mathfrak{g}$  is a simple Lie algebra of type  $G_2$ . The explicit formulas related to the Weyl basis of simple Lie algebras of infinite series appear in [1, Lemma 6.5], [5, Lemma 3.5], [6, Theorem 4.3] and [7].

The Weyl basis of  $\mathfrak{g}$  of the type  $G_2$  consists of the polynomials

$$(1.1) \quad x_1, x_2, [x_1, x_2], [[x_1, x_2], x_2], [[[x_1, x_2], x_2], x_2], [[[[x_1, x_2], x_2], x_2], x_1],$$

that correspond to the positive roots

$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2,$$

see [10, Chapter VI, §4] or [4, Chapter IV, §3, XVII]. These polynomials define some elements of  $U_q^+(\mathfrak{g})$  if the Lie operation is replaced with the skew brackets. The coproduct of all of them, except the last one, may be found by the general formula proved in Proposition 2.7:

$$\Delta([x_1 x_2^n]) = [x_1 x_2^n] \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} g_1 g_2^{n-k} x_2^k \otimes [x_1 x_2^{n-k}],$$

where

$$\alpha_k^{(n)} = \begin{bmatrix} n \\ k \end{bmatrix}_{p_{22}} \cdot \prod_{s=n-k}^{n-1} (1 - p_{12} p_{21} p_{22}^s),$$

and  $p_{ij}$  are the quantization parameters that define the commutation rules between the group-like elements and the generators:  $x_i g_j = p_{ij} g_j x_i$ . This formula takes more elegant form if we consider proportional elements

$$\{x^n\} = \frac{x^n}{[n]_q!}, \quad \{xy^n\} = \frac{[\dots [x, y], y], \dots, y]}{[n]_q! \cdot \prod_{s=0}^{n-1} (1 - p_{12} p_{21} p_{22}^s)}.$$

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For these elements, the coproduct obeys the following form:

$$\Delta(\{x_2^n\}) = \sum_{k=0}^n g_2^{n-k} \{x_2^k\} \otimes \{x_2^{n-k}\},$$

$$\Delta(\{x_1 x_2^n\}) = \{x_1 x_2^n\} \otimes 1 + \sum_{k=0}^n g_1 g_2^{n-k} \{x_2^k\} \otimes \{x_1 x_2^{n-k}\}.$$

Clearly, the above formulas make sense only if  $[k]_q \neq 0$ ,  $p_{12}p_{21} \neq q^{1-k}$ ,  $q = p_{22}$  for all  $k$ ,  $1 \leq k \leq n$  because otherwise the elements  $\{x_2^n\}$  and  $\{x_1 x_2^n\}$  are undefined. A similar formula is valid for the elements with opposite alignment of brackets

$$\Delta(\{x_2^n x_1\}) = \{x_2^n x_1\} \otimes 1 + \sum_{k=0}^n g_2^k \{x_2^{n-k} x_1\} \otimes \{x_2^k\},$$

where by definition

$$\{x_2^n x_1\} = \frac{[x_2, [x_2, \dots [x_2, x_1] \dots]]}{[n]_q! \cdot \prod_{s=0}^{n-1} (1 - p_{12}p_{21}p_{22}^s)}.$$

The coproduct of the remaining in (1.1) element has no elegant form. Nevertheless, if we replace it with

$$\{x_1 x_2^3 x_1\} \stackrel{df}{=} p_{21}^2 \frac{q^3 + q^2}{1 - q^3} \{x_1 x_2\} \{x_1 x_2^2\} + p_{21} \frac{[4]_q - 2}{1 - q^3} \{x_1 x_2^2\} \{x_1 x_2\} + \{x_1 x_2^3\} x_1,$$

then the ordered set

$$x_2 < \{x_1 x_2^3\} < \{x_1 x_2^2\} < \{x_1 x_2^3 x_1\} < \{x_1 x_2\} < x_1$$

forms a set of PBW generators of the algebra  $U_q^+(\mathfrak{g})$  over  $G$  (Prorosition 4.1), and a harmonic formula is valid (Theorem 4.2):

$$\Delta(\{x_1 x_2^3 x_1\}) = \{x_1 x_2^3 x_1\} \otimes 1 + g_1^2 g_2^3 \{x_1 x_2^3 x_1\} + \sum_{k=0}^3 g_1 g_2^k \{x_2^{3-k} x_1\} \otimes \{x_1 x_2^k\}.$$

Of course, the set of PBW generators for  $U_q(\mathfrak{g})$  is the union of that sets for positive and negative quantum Borel subalgebras. By this reason we consider only the positive quantum Borel subalgebra  $U_q^+(\mathfrak{g})$ .

## 2. COPRODUCT OF SERRE POLYNOMIALS

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of quantum variables; that is, associated with each  $x_i$  are an element  $g_i$  of a fixed Abelian group  $G$  and a character  $\chi^i : G \rightarrow \mathbf{k}^*$ . For every word  $w$  in  $X$  let  $g_w$  or  $\text{gr}(w)$  denote an element of  $G$  that appears from  $w$  by replacing each  $x_i$  with  $g_i$ . In the same way  $\chi^w$  denotes a character that appears from  $w$  by replacing each  $x_i$  with  $\chi^i$ .

Let  $G\langle X \rangle$  denote the skew group algebra generated by  $G$  and  $\mathbf{k}\langle X \rangle$  with the commutation rules  $x_i g = \chi^i(g) g x_i$ , or equivalently  $w g = \chi^w(g) g w$ , where  $w$  is an arbitrary word in  $X$ . If  $u, v$  are homogeneous polynomials in each  $x_i$ ,  $1 \leq i \leq n$ , then the skew brackets are defined by the formula

$$(2.1) \quad [u, v] = uv - \chi^u(g_v) v u.$$

We use the notation  $\chi^u(g_v) = p_{uv} = p(u, v)$ . The form  $p(-, -)$  is bimultiplicative:

$$(2.2) \quad p(u, vt) = p(u, v)p(u, t), \quad p(ut, v) = p(u, v)p(t, v).$$

In particular  $p(-, -)$  is completely defined by  $n^2$  parameters  $p_{ij} = \chi^i(g_j)$ .

The algebra  $G\langle X \rangle$  has a Hopf algebra structure given by the comultiplications on the generators:

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad 1 \leq i \leq n, \quad \Delta(g) = g \otimes g, \quad g \in G.$$

The so-called  $q$ -Serre (noncommutative) polynomials,

$$[\dots [x_i, \underbrace{x_j, \dots, x_j}_m], \dots, x_j] \stackrel{df}{=} [x_i x_j^m], \quad 1 \leq i \neq j \leq n,$$

and

$$[\underbrace{x_j, [x_j, \dots [x_j, x_i] \dots]}_m] \stackrel{df}{=} [x_j^m x_i], \quad 1 \leq i \neq j \leq n,$$

are important as the defining relations of the quantizations  $U_q^+(\mathfrak{g})$ . To find the coproduct of that polynomials, we recall the notations of the  $q$ -combinatoric.

If  $q$  is a fixed parameter, then  $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$  and  $[n]_q! = \prod_{k=1}^n [k]_q$ . The Gauss polynomials are defined as  $q$ -binomial coefficients,

$$(2.3) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!},$$

that satisfy two  $q$ -Pascal identities

$$(2.4) \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \cdot \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \cdot q^{n-k+1} + \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

If  $x$  and  $y$  are variables subject to the relation  $yx = qxy$ , then  $q$ -Neuton binomial formula is valid

$$(2.5) \quad (x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k.$$

For example, if we put  $x = g_2 \otimes x_2$ ,  $y = x_2 \otimes 1$ , then

$$yx = x_2 g_2 \otimes x_2 = p_{22} g_2 x_2 \otimes x_2 = qxy, \quad q = p_{22},$$

and the  $q$ -Neuton binomial formula implies

$$(2.6) \quad \Delta(x_2^n) = (x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q g_2^{n-k} x_2^k \otimes x_2^{n-k}.$$

**Proposition 2.1.** *The following explicit coproduct formula is valid:*

$$(2.7) \quad \Delta([x_1 x_2^n]) = [x_1 x_2^n] \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} g_1 g_2^{n-k} x_2^k \otimes [x_1 x_2^{n-k}],$$

where

$$(2.8) \quad \alpha_k^{(n)} = \begin{bmatrix} n \\ k \end{bmatrix}_{p_{22}} \cdot \prod_{s=n-k}^{n-1} (1 - p_{12} p_{21} p_{22}^s).$$

*Proof.* We shall use induction on  $n$ . If  $n = 0$ , then the equality reduces to  $\Delta(x_1) = x_1 \otimes 1 + g_1 \otimes x_1$ , whereas  $\alpha_0^{(0)} = 1$ . Moreover, it is clear that  $\alpha_0^{(n)} = 1$  for all  $n$ . We have,

$$(2.9) \quad \Delta([x_1 x_2^n]) \cdot (x_2 \otimes 1) = [x_1 x_2^n] x_2 \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} g_1 g_2^{n-k} x_2^{k+1} \otimes [x_1 x_2^{n-k}],$$

$$(2.10) \quad \Delta([x_1 x_2^n]) \cdot (g_2 \otimes x_2) = [x_1 x_2^n] g_2 \otimes x_2 + \sum_{k=0}^n \alpha_k^{(n)} g_1 g_2^{n-k} x_2^k g_2 \otimes [x_1 x_2^{n-k}] x_2,$$

$$(2.11) \quad (x_2 \otimes 1) \cdot \Delta([x_1 x_2^n]) = x_2 [x_1 x_2^n] \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} x_2 g_1 g_2^{n-k} x_2^k \otimes [x_1 x_2^{n-k}],$$

$$(2.12) \quad (g_2 \otimes x_2) \cdot \Delta([x_1 x_2^n]) = g_2 [x_1 x_2^n] \otimes x_2 + \sum_{k=0}^n \alpha_k^{(n)} g_1 g_2^{n-k+1} x_2^k \otimes x_2 [x_1 x_2^{n-k}].$$

In the second and third relations we may move the group-like factors to the left:

$$[x_1 x_2^n] g_2 = p_{12} p_{22}^n g_2 [x_1 x_2^n], \quad x_2^k g_2 = p_{22}^k g_2 x_2^k, \quad x_2 g_1 g_2^{n-k} x_2^k = p_{21} p_{22}^{n-k} g_1 g_2^{n-k+1} x_2^{k+1}.$$

Using all that relations, we develop the coproduct of

$$[x_1 x_2^{n+1}] = [x_1 x_2^n] x_2 - p_{12} p_{22}^n x_2 [x_1 x_2^n]$$

taking into account that  $\Delta(x_2) = x_2 \otimes 1 + g_2 \otimes x_2$ . The sums of (2.9) and (2.11) provide the tensors

$$[x_1 x_2^{n+1}] \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} (1 - p_{12} p_{21} p_{22}^{2n-k}) g_1 g_2^{n-k} x_2^{k+1} \otimes [x_1 x_2^{n-k}],$$

whereas the sums of (2.10) and (2.12) produce the following ones:

$$\sum_{k=0}^n \alpha_k^{(n)} p_{22}^k g_1 g_2^{n-k+1} x_2^k \otimes [x_1 x_2^{n-k+1}].$$

The first term of (2.10) cancels with the first term of (2.12). Finally, we arrive to the formula (2.7) with  $n \leftarrow n+1$  and coefficients

$$(2.13) \quad \alpha_k^{(n+1)} = \alpha_{k-1}^{(n)} (1 - p_{12} p_{21} p_{22}^{2n-k+1}) + \alpha_k^{(n)} p_{22}^k, \quad k \geq 1, \quad \alpha_0^{(n+1)} = 1.$$

To prove the coproduct formula (2.7), it remains to check that values (2.8) satisfy the above recurrence relations.

To this end, we shall check the equality of the following two polynomials in commutative variables  $\lambda, q$ :

$$(2.14) \quad \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \cdot (1 - \lambda q^n) = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \cdot (1 - \lambda q^{2n-k+1}) + \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot (1 - \lambda q^{n-k}) \cdot q^k.$$

If  $\lambda = 0$ , then the equality reduces to the first  $q$ -Pascal identity (2.4). Let us compare the coefficients at  $\lambda$ ,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \cdot q^n = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \cdot q^{2n-k+1} + \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot q^{n-k} \cdot q^k.$$

This equality differs from the second  $q$ -Pascal identity (2.4) just by a common factor  $q^n$ . Hence, the equality (2.14) is valid.

If we multiply both sides of (2.14) by  $\prod_{s=n-k+1}^{n-1} (1 - \lambda q^s)$  and next replace the variables  $q \leftarrow p_{22}$ ,  $\lambda \leftarrow p_{12} p_{21}$ , then we obtain precisely (2.13) for values (2.8).  $\square$

One may eliminate all coefficients in these coproduct formulas replacing the elements by some scalar multiples of them. Let us put

$$(2.15) \quad \{x^n\} = \frac{x^n}{[n]_q!}, \quad \{xy^n\} = \frac{[xy^n]}{[n]_q! \cdot \prod_{s=0}^{n-1} (1 - p_{12} p_{21} p_{22}^s)}.$$

Then the coproduct formulas obey more elegant form:

$$(2.16) \quad \Delta(\{x_2^n\}) = \sum_{k=0}^n g_2^{n-k} \{x_2^k\} \otimes \{x_2^{n-k}\},$$

$$(2.17) \quad \Delta(\{x_1 x_2^n\}) = \{x_1 x_2^n\} \otimes 1 + \sum_{k=0}^n g_1 g_2^{n-k} \{x_2^k\} \otimes \{x_1 x_2^{n-k}\}.$$

Of course, the above formulas make sense only if  $[k]_q \neq 0$ ,  $p_{12}p_{21} \neq q^{1-k}$ ,  $q = p_{22}$  for all  $k$ ,  $1 \leq k \leq n$  because otherwise the elements  $\{x_2^n\}$  and  $\{x_1 x_2^n\}$  are undefined.

In perfect analogy one may develop a coproduct formula for the elements

$$\underbrace{[x_2, [x_2, \dots [x_2, x_1] \dots]]}_n \stackrel{df}{=} [x_2^n x_1] :$$

$$(2.18) \quad \Delta([x_2^n x_1]) = [x_2^n x_1] \otimes 1 + \sum_{k=0}^n \alpha_k^{(n)} g_2^k [x_2^{n-k} x_1] \otimes x_2^k.$$

If we define

$$\{x_2^n x_1\} = \frac{[x_2^n x_1]}{[n]_q! \cdot \prod_{s=0}^{n-1} (1 - p_{12}p_{21}p_{22}^s)},$$

then all coefficients disappear:

$$(2.19) \quad \Delta(\{x_2^n x_1\}) = \{x_2^n x_1\} \otimes 1 + \sum_{k=0}^n g_2^k \{x_2^{n-k} x_1\} \otimes \{x_2^k\}.$$

Below we show an alternative way how to develop these coproduct formulas using the shuffle representation.

### 3. SHUFFLE REPRESENTATION

The tensor space  $T(W)$  of the linear space  $W$  spanned by the set of quantum variables  $X = \{x_1, x_2, \dots, x_n\}$  has a structure of a braided Hopf algebra  $Sh(W)$  with a braiding  $\tau(u \otimes v) = p(v, u)^{-1} v \otimes u$ . We shall denote the tensors  $z_1 \otimes z_2 \otimes \dots \otimes z_m$ ,  $z_i \in X$  considered as elements of  $Sh(W)$  by  $(z_1 z_2 \dots z_n)$  and call them *comonomials*. By definition the product on  $T(W)$  is the so called *shuffle product*:

$$(3.1) \quad (u)(v) = \sum_{\substack{u=u_1 \dots u_\ell, v=v_1 \dots v_\ell \\ 1 \leq \ell \leq \max\{\deg(u), \deg(v)\}}} \left( \prod_{i < j} p_{v_i u_j}^{-1} \right) (u_1 v_1 \dots u_\ell v_\ell),$$

whereas the coproduct  $\Delta^b : Sh(W) \rightarrow Sh(W) \underline{\otimes} Sh(W)$  is defined by

$$(3.2) \quad \Delta^b((u)) = \sum_{u=u_1 u_2} (u_1) \otimes (u_2).$$

The formula of the shuffle product is easier when one of the comonomials has the length one:

$$(3.3) \quad (w)(x_i) = \sum_{uv=w} p(x_i, v)^{-1} \cdot (ux_i v), \quad (x_i)(w) = \sum_{uv=w} p(u, x_i)^{-1} \cdot (ux_i v).$$

From this equality, we deduce that

$$(3.4) \quad [(w), (x_i)] = \sum_{uv=w} (p(x_i, v)^{-1} - p(v, x_i)) \cdot (ux_i v),$$

and

$$(3.5) \quad [(x_i), (w)] = \sum_{uv=w} (p(u, x_i)^{-1} - p(x_i, u)) \cdot (ux_i v).$$

The free algebra  $\mathbf{k}\langle X \rangle$  considered as a subalgebra of  $G\langle X \rangle$  becomes a braided Hopf algebra if we define a braided coproduct  $\Delta^b$  as follows:

$$(3.6) \quad \Delta^b(u) = \sum_{(u)} u^{(1)} \text{gr}(u^{(2)})^{-1} \underline{\otimes} u^{(2)}, \text{ where } \Delta(u) = \sum_{(u)} u^{(1)} \otimes u^{(2)}.$$

The map  $\Omega : x_i \rightarrow (x_i)$  defines a homomorphism of the braided Hopf algebra  $\mathbf{k}\langle X \rangle$  into the braided Hopf algebra  $Sh(W)$ . If  $p_{ij}$  are algebraically independent parameters, then  $\Omega$  is an isomorphism. Otherwise the kernel of  $\Omega$  is the largest Hopf ideal in  $\mathbf{k}\langle X \rangle^{(2)}$ , where  $\mathbf{k}\langle X \rangle^{(2)}$  is the ideal of  $\mathbf{k}\langle X \rangle$ , generated by  $x_i x_j$ ,  $1 \leq i, j \leq n$ . The image of  $\Omega$  is the so-called Nichols algebra of the braided space  $W$ . See details in P. Schauenberg [11], M. Rosso [9], M. Takeuchi [12], D. Flores de Chela and J.A. Green [3], N. Andruskiewitsch, H.-J. Schneider [1].

The homomorphism  $\Omega$  is extremely useful for calculating the coproduct due to (3.6) and (3.2). In this way, we may find alternative proof of the coproduct formulas (2.17) and (2.19).

**Lemma 3.1.** *If  $[n]_q! \neq 0$ , then*

$$\Omega(\{x_2^n\}) = q^{\frac{n(1-n)}{2}}(x_2^n), \quad q = p_{22}.$$

*Otherwise  $\Omega(x_2^n) = 0$ .*

*Proof.* By induction on  $n$  we may prove the explicit formula

$$(3.7) \quad \Omega(x_2^n) = [n]_{q^{-1}}!(x_2^n).$$

Indeed, if  $n = 1$ , then there is nothing to prove. Using (3.3), we have

$$\Omega(x_2^{n+1}) = [n]_{q^{-1}}!(x_2^n)(x_2) = [n]_{q^{-1}}!(1 + q^{-1} + \dots + q^{-n})(x_2^{n+1}) = [n+1]_{q^{-1}}!(x_2^{n+1}).$$

Because  $[k]_{q^{-1}} = q^{-(k-1)} \cdot [k]_q$ , and  $-1 - 2 - \dots - n = \frac{(n+1)(1-(n+1))}{2}$ , the lemma is proven.  $\square$

**Lemma 3.2.** *If  $[k]_q \neq 0$ ,  $p_{12}p_{21} \neq q^{1-k}$ ,  $q = p_{22}$ ,  $1 \leq k \leq n$ , then*

$$\Omega(\{x_1 x_2^n\}) = p_{21}^{-n} q^{\frac{n(1-n)}{2}}(x_2^n x_1),$$

*and*

$$\Omega(\{x_2^n x_1\}) = p_{12}^{-n} q^{\frac{n(1-n)}{2}}(x_1 x_2^n).$$

*Otherwise  $\Omega([x_1 x_2^n]) = \Omega([x_2^n x_1]) = 0$ .*

*Proof.* By induction on  $n$  we shall prove the explicit formulas

$$(3.8) \quad \Omega([x_1 x_2^n]) = [n]_{q^{-1}}! \cdot p_{21}^{-n} \prod_{s=0}^{n-1} (1 - p_{12}p_{21}q^s) \cdot (x_2^n x_1),$$

*and*

$$(3.9) \quad \Omega([x_2^n x_1]) = [n]_{q^{-1}}! \cdot p_{12}^{-n} \prod_{s=0}^{n-1} (1 - p_{12}p_{21}q^s) \cdot (x_1 x_2^n).$$

If  $n = 0$ , the equalities are evident. Using (3.4), we have

$$\begin{aligned} [(x_2^n x_1), (x_2)] &= \sum_{uv=x_2^n x_1} (p(x_2, v)^{-1} - p(v, x_2)) \cdot (ux_2 v) \\ &= \sum_{k=0}^n (p_{21}^{-1} q^{-k} - p_{12} q^k) \cdot (x_2^{n+1} x_1) = (p_{21}^{-1} [n]_{q^{-1}} - p_{12} [n]_q) \cdot (x_2^{n+1} x_1) \\ &= (1 - p_{12} p_{21} q^n) p_{21}^{-1} [n]_{q^{-1}} \cdot (x_2^{n+1} x_1), \end{aligned}$$

which completes the induction step. Similarly, using (3.5), we have

$$\begin{aligned} [(x_2), (x_1 x_2^n)] &= \sum_{uv=x_1 x_2^n} (p(u, x_2)^{-1} - p(x_2, u)) \cdot (ux_2 v) \\ &= \sum_{k=0}^n (p_{12}^{-1} q^{-k} - p_{21} q^k) \cdot (x_1 x_2^{n+1}) = (p_{12}^{-1} [n]_{q^{-1}} - p_{21} [n]_q) \cdot (x_1 x_2^{n+1}) \\ &= (1 - p_{12} p_{21} q^n) p_{12}^{-1} [n]_{q^{-1}} \cdot (x_1 x_2^{n+1}), \end{aligned}$$

which proves the second equality. Since  $[n]_{q^{-1}}! = q^{\frac{n(1-n)}{2}} \cdot [n]_q!$ , the lemma is proven.  $\square$

The coproduct formula (2.19) follows from the proven lemmas in the following way.

$$\begin{aligned} (\Omega \otimes \Omega) \Delta^b(\{x_2^n x_1\}) &= \Delta^b(\Omega(\{x_2^n x_1\})) = p_{12}^{-n} q^{\frac{n(1-n)}{2}} \Delta^b((x_1 x_2^n)) \\ &= p_{12}^{-n} q^{\frac{n(1-n)}{2}} \left( 1 \underline{\otimes} (x_1 x_2^n) + \sum_{k=0}^n (x_1 x_2^{n-k}) \underline{\otimes} (x_2^k) \right) \\ &= 1 \underline{\otimes} \Omega\{x_2^n x_1\} + p_{12}^{-n} q^{\frac{n(1-n)}{2}} \sum_{k=0}^n p_{12}^{-k} q^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \Omega\{x_2^{n-k} x_1\} \underline{\otimes} \Omega\{x_2^k\} = \\ &= (\Omega \otimes \Omega) \left( 1 \underline{\otimes} \{x_2^n x_1\} + \sum_{k=0}^n p_{12}^{-k} q^{k(k-n)} \{x_2^{n-k} x_1\} \underline{\otimes} \{x_2^k\} \right). \end{aligned}$$

Considering that  $p_{ij}$  are algebraically independent parameters, we have

$$\Delta^b(\{x_2^n x_1\}) = 1 \underline{\otimes} \{x_2^n x_1\} + \sum_{k=0}^n p_{12}^{-k} \cdot q^{k(k-n)} \cdot \{x_2^{n-k} x_1\} \underline{\otimes} \{x_2^k\}.$$

Due to (3.6), we may write  $u^{(1)} = u_b^{(1)} \text{gr}(u^{(2)})$  and  $u^{(2)} = u_b^{(2)}$ . We have  $\text{gr}(\{x_2^k\}) = g_2^k$ , and  $\{x_2^{n-k} x_1\} g_2^k = p_{12}^k \cdot q^{k(n-k)} g_2^k \{x_2^{n-k} x_1\}$ . Hence the component  $u^{(1)} \otimes u^{(2)}$  is precisely  $g_2^k \{x_2^{n-k} x_1\} \otimes \{x_2^k\}$ . Of course, if the proven formula is valid for free parameters, it remains valid for arbitrary parameters provided that  $\{x_2^k\}$  and  $\{x_2^{n-k} x_1\}$ ,  $0 \leq k \leq n$  are defined.

**Lemma 3.3.** *If  $p_{11} = q^3$ ,  $p_{22} = q$ ,  $p_{12} p_{21} = q^{-3} \neq 1$ , then in the shuffle algebra the following decomposition is valid:*

$$(x_1 x_2^3 x_1) = p_{12} \frac{q^2 + q}{1 - q^3} \cdot (x_2 x_1)(x_2^2 x_1) + q^2 p_{12}^2 \frac{[4]_q - 2}{1 - q^3} \cdot (x_2^2 x_1)(x_2 x_1) + q^3 p_{12}^3 \cdot (x_2^3 x_1)(x_1).$$

*Proof.* We have

$$\begin{aligned}
(x_2 x_1)(x_2 x_2 x_1) &= \\
& (x_2 x_1 x_2 x_2 x_1) + p_{21}^{-1}(x_2 x_2 x_1 x_2 x_1) + p_{21}^{-2}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-2} p_{11}^{-1}(x_2 x_2 x_2 x_1 x_1) \\
& + p_{21}^{-1} p_{22}^{-1}(x_2 x_2 x_1 x_2 x_1) + p_{21}^{-2} p_{22}^{-1}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-1} p_{11}^{-1} p_{22}^{-1}(x_2 x_2 x_2 x_1 x_1) \\
& + p_{21}^{-2} p_{22}^{-2}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-2} p_{11}^{-1} p_{22}^{-2}(x_2 x_2 x_2 x_1 x_1) \\
& + p_{21}^{-2} p_{11}^{-1} p_{22}^{-1} p_{12}^{-1}(x_2 x_2 x_1 x_2 x_1) \\
& = (x_2 x_1 x_2^2 x_1) + p_{21}^{-1}(1 + p_{22}^{-1} + p_{22}^{-2})(x_2^2 x_1 x_2 x_1) \\
& + p_{21}^{-2}(1 + p_{22}^{-1} + p_{22}^{-2})(1 + p_{11}^{-1})(x_2^3 x_1^2).
\end{aligned}$$

$$\begin{aligned}
(x_2 x_2 x_1)(x_2 x_1) &= \\
& (x_2 x_2 x_1 x_2 x_1) + p_{21}^{-1}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-1} p_{22}^{-1}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-1} p_{22}^{-2}(x_2 x_2 x_2 x_1 x_1) \\
& + p_{21}^{-1} p_{11}^{-1}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-1} p_{22}^{-1} p_{11}^{-1}(x_2 x_2 x_2 x_1 x_1) + p_{21}^{-1} p_{22}^{-2} p_{11}^{-1}(x_2 x_2 x_2 x_1 x_1) \\
& + p_{21}^{-1} p_{22}^{-1} p_{11}^{-1} p_{12}^{-1}(x_2 x_2 x_1 x_2 x_1) + p_{21}^{-1} p_{22}^{-2} p_{11}^{-1} p_{12}^{-1}(x_2 x_2 x_1 x_2 x_1) \\
& + p_{21}^{-1} p_{22}^{-1} p_{11}^{-1} p_{12}^{-2}(x_2 x_1 x_2 x_2 x_1) \\
& = p_{12}^{-1} p_{22}^{-2}(x_2 x_1 x_2^2 x_1) + (1 + p_{22}^{-1} + p_{22}^{-2})(x_2^2 x_1 x_2 x_1) \\
& + p_{21}^{-1}(1 + p_{22}^{-1} + p_{22}^{-2})(1 + p_{11}^{-1})(x_2^3 x_1^2).
\end{aligned}$$

$$\begin{aligned}
(x_2 x_2 x_2 x_1)(x_1) &= p_{11}^{-1} p_{12}^{-2}(x_2 x_1 x_2^2 x_1) + p_{11}^{-1} p_{12}^{-1}(x_2^2 x_1 x_2 x_1) \\
& + (1 + p_{11}^{-1})(x_2^3 x_1^2) + p_{11}^{-1} p_{12}^{-3}(x_1 x_2^3 x_1).
\end{aligned}$$

Taking into account relations  $p_{11} = q^3$ ,  $p_{22} = q$ ,  $p_{21} = q^{-3} p_{12}^{-1}$ ,  $1 + p_{22}^{-1} + p_{22}^{-2} = q^{-2}[3]_q$ , we obtain

$$\begin{aligned}
(x_2 x_1)(x_2^2 x_1) &= (x_2 x_1 x_2^2 x_1) + q[3]_q p_{12}(x_2^2 x_1 x_2 x_1) + (q^3 + 1)q[3]_q p_{12}^2(x_2^3 x_1^2) \\
(x_2^2 x_1)(x_2 x_1) &= q^{-2} p_{12}^{-1}(x_2 x_1 x_2^2 x_1) + q^{-2}[3]_q(x_2^2 x_1 x_2 x_1) + (q^3 + 1)q^{-2}[3]_q p_{12}(x_2^3 x_1^2) \\
(x_2^3 x_1)(x_1) - q^{-3} p_{12}^{-3}(x_1 x_2^3 x_1) &= \\
& = q^{-3} p_{12}^{-2}(x_2 x_1 x_2^2 x_1) + q^{-3} p_{12}^{-1}(x_2^2 x_1 x_2 x_1) + (q^3 + 1)q^{-3}(x_2^3 x_1^2).
\end{aligned}$$

The determinant of the  $3 \times 3$  matrix of the right-hand side coefficients is zero because the last two columns are proportional with respect to the factor  $(q^3 + 1)p_{12}$ . Therefore, the shuffles from the left-hand side are linearly dependent. In which case the minors

$$\begin{vmatrix} 1 & q[3]_q p_{12} \\ q^{-2} p_{12}^{-1} & q^{-2}[3]_q \end{vmatrix} = (q^{-2} - q^{-1})[3]_q; \quad \begin{vmatrix} q^{-2} p_{12}^{-1} & q^{-2}[3]_q \\ q^{-3} p_{12}^{-2} & q^{-3} p_{12}^{-1} \end{vmatrix} = -q^{-4}(1 + q)p_{12}^{-2}$$

are nonzero; that is, the first two shuffles are linearly independent as well as the second one and the last one are. Thus, the last left-hand side shuffle is a linear combination of the first and second ones. Of course, it is easy to find the explicit values of the coefficients  $\alpha$  and  $\beta$  resolving the system of equations

$$\begin{cases} \alpha + q^{-2} p_{12}^{-1} \beta = q^{-3} p_{12}^{-2} \\ q[3]_q p_{12} \alpha + q^{-2}[3]_q \beta = q^{-3} p_{12}^{-1}, \end{cases}$$

$$\alpha = -q^{-2} p_{12}^{-2} \frac{q+1}{1-q^3}; \quad \beta = q^{-1} p_{12}^{-1} \frac{2-[4]_q}{1-q^3}.$$



To find the coefficients of the required decomposition, it remains to multiply  $\alpha$  and  $\beta$  by  $-q^3 p_{12}^3$ .  $\square$

#### 4. COPRODUCT FORMULA

By definition the quantum Borel algebra  $U_q^+(\mathfrak{g})$  related to the simple Lie algebra of type  $G_2$  is a homomorphic image of  $G\langle x_1, x_2 \rangle$  subject to quantum Serre relations

$$[x_1 x_2^4] = 0, \quad [x_1^2 x_2] = 0$$

provided that the parameters of quantization satisfy

$$p_{11} = p_{12}^{-1} p_{21}^{-1} = p_{22}^3 \neq 1, \quad p_{22} \neq -1.$$

Of course, this is a two-parameter family of Hopf algebras. As above, we put  $q = p_{22}$ . Coproduct formulas (2.7) with  $n = 4$  and (2.18) with  $n = 2$  imply that the defining relations are skew-primitive polynomials in  $G\langle x_1, x_2 \rangle$ . Therefore  $U_q^+(\mathfrak{g})$  keeps the Hopf algebra structure.

The subalgebra  $A$  of  $U_q^+(\mathfrak{g})$  generated by  $x_1, x_2$  has a structure of braided Hopf algebra if we define the braided coproduct by the same relation (3.6). Equation (3.8) with  $n = 4$  and equation (3.9) with  $n = 2$  show that  $\Omega([x_1 x_2^4]) = 0$  and  $\Omega([x_1^2 x_2]) = 0$ . Therefore the homomorphism  $\Omega$  induces a homomorphism of braided Hopf algebras

$$\bar{\Omega} : A \rightarrow Sh(W).$$

It is well-known that if  $q$  is not a root of unity, then this is an isomorphism. We define a new element of  $A$  as follows:

$$\{x_1 x_2^3 x_1\} \stackrel{df}{=} p_{21}^2 \frac{q^3 + q^2}{1 - q^3} \{x_1 x_2\} \{x_1 x_2^2\} + p_{21} \frac{[4]_q - 2}{1 - q^3} \{x_1 x_2^2\} \{x_1 x_2\} + \{x_1 x_2^3\} x_1.$$

**Proposition 4.1.** *The ordered set*

$$x_2 < \{x_1 x_2^3\} < \{x_1 x_2^2\} < \{x_1 x_2^3 x_1\} < \{x_1 x_2\} < x_1$$

*forms a set of PBW generators of the algebra  $U_q^+(\mathfrak{g})$  over  $G$ .*

*Proof.* In [8] and independently in [2], it is shown that the ordered set

$$x_2 < [x_1 x_2^3] < [x_1 x_2^2] < [[x_1 x_2], [x_1 x_2^2]] < [x_1 x_2] < x_1$$

forms a set of PBW generators of the algebra  $U_q^+(\mathfrak{g})$  over  $G$ . Of course, this implies that

$$x_2 < \{x_1 x_2^3\} < \{x_1 x_2^2\} < [\{x_1 x_2\}, \{x_1 x_2^2\}] < \{x_1 x_2\} < x_1$$

is also a set of PBW generators. By definition the element  $\{x_1 x_2^3 x_1\}$  has the form

$$\{x_1 x_2^3 x_1\} = \alpha \{x_1 x_2\} \{x_1 x_2^2\} + \beta \{x_1 x_2^2\} \{x_1 x_2\} + \{x_1 x_2^3\} x_1,$$

where  $\alpha \neq 0$ . Using evident formula  $uv = [u, v] + p(u, v)vu$ , we obtain the decomposition of  $\{x_1 x_2^3 x_1\}$  in the above PBW basis:

$$\alpha [\{x_1 x_2\}, \{x_1 x_2^2\}] + \gamma \{x_1 x_2^2\} \{x_1 x_2\} + \{x_1 x_2^3\} x_1.$$

In this decomposition  $[\{x_1 x_2\}, \{x_1 x_2^2\}]$  is the leading term. Since  $\alpha \neq 0$ , it follows that in the set of PBW generators we may replace  $[\{x_1 x_2\}, \{x_1 x_2^2\}]$  with  $\{x_1 x_2^3 x_1\}$ .  $\square$

**Theorem 4.2.** *The following coproduct formula is valid*

$$\Delta(\{x_1 x_2^3 x_1\}) = \{x_1 x_2^3 x_1\} \otimes 1 + g_1^2 g_2^3 \otimes \{x_1 x_2^3 x_1\} + \sum_{k=0}^3 g_1 g_2^k \{x_2^{3-k} x_1\} \otimes \{x_1 x_2^k\}.$$

*Proof.* By Lemma 3.2, we have

$$\bar{\Omega}(\{x_1 x_2\}) = p_{21}^{-1}(x_2 x_1), \quad \bar{\Omega}(\{x_1 x_2^2\}) = p_{21}^{-2} q^{-1}(x_2^2 x_1), \quad \bar{\Omega}(\{x_1 x_2^3\}) = p_{21}^{-3} q^{-3}(x_2^3 x_1).$$

Using these equalities and Lemma 3.3, we obtain

$$\begin{aligned} \bar{\Omega}(\{x_1 x_2^3 x_1\}) &= p_{21}^2 \frac{q^3 + q^2}{1 - q^3} \cdot p_{21}^{-1} \cdot p_{21}^{-2} q^{-1}(x_2 x_1)(x_2^2 x_1) \\ &\quad + p_{21} \frac{[4]q - 2}{1 - q^3} \cdot p_{21}^{-1} \cdot p_{21}^{-2} q^{-1}(x_2^2 x_1)(x_2 x_1) + p_{21}^{-3} q^{-3}(x_2^3 x_1)(x_1) = q^3(x_1 x_2^3 x_1) \end{aligned}$$

because  $p_{21}^{-1} q^{-1} = q^3 \cdot p_{12}$ ,  $p_{21}^{-2} q^{-1} = q^3 \cdot q^2 p_{12}^2$ , and  $p_{21}^{-3} q^{-3} = q^3 \cdot q^3 p_{12}^3$ . We have,

$$\begin{aligned} (\bar{\Omega} \otimes \bar{\Omega}) \Delta^b(\{x_1 x_2^3 x_1\}) &= \Delta^b(\bar{\Omega}(\{x_1 x_2^3 x_1\})) = \Delta^b(q^3(x_1 x_2^3 x_1)) \\ &= q^3(x_1 x_2^3 x_1) \otimes 1 + 1 \otimes q^3(x_1 x_2^3 x_1) + \sum_{k=0}^3 q^3(x_1 x_2^{3-k}) \otimes (x_2^k x_1). \end{aligned}$$

Lemma 3.2 implies

$$(x_1 x_2^{3-k}) = p_{12}^{3-k} q^{\frac{(3-k)(3-k-1)}{2}} \bar{\Omega}(\{x_2^{3-k} x_1\}), \quad (x_2^k x_1) = p_{21}^k q^{\frac{k(k-1)}{2}} \bar{\Omega}(\{x_1 x_2^k\}).$$

By this reason, the tensor under the sum equals

$$p_{12}^{3-k} p_{21}^k q^{6-3k+k^2} \cdot \bar{\Omega} \otimes \bar{\Omega}(\{x_2^{3-k} x_1\} \otimes \{x_1 x_2^k\}).$$

If  $q$  is a free parameter, then  $\bar{\Omega}$  is an isomorphism, and in  $A$ , we have

$$\Delta^b(\{x_1 x_2^3 x_1\}) = \{x_1 x_2^3 x_1\} \otimes 1 + 1 \otimes \{x_1 x_2^3 x_1\} + \sum_{k=0}^3 \tau_k g_1 g_2^k \{x_2^{3-k} x_1\} \otimes \{x_1 x_2^k\},$$

where  $\tau_k = p_{12}^{3-k} p_{21}^k q^{6-3k+k^2}$ . Due to (3.6), we may write  $u^{(1)} = u_b^{(1)} \text{gr}(u^{(2)})$  and  $u^{(2)} = u_b^{(2)}$ . Since  $\text{gr}(\{x_1 x_2^k\}) = g_1 g_2^k$ , and

$$\{x_2^{3-k} x_1\} g_1 g_2^k = \mu_k \cdot g_1 g_2^k \{x_2^{3-k} x_1\}$$

with  $\mu_k = p_{11} p_{12}^k p_{21}^{3-k} p_{22}^{(3-k)k}$ , it follows that the component  $u^{(1)} \otimes u^{(2)}$  equals  $\tau_k \mu_k g_1 g_2^k \{x_2^{3-k} x_1\} \otimes \{x_1 x_2^k\}$ . It remains to check that  $\tau_k \mu_k = 1$ :

$$p_{12}^{3-k} p_{21}^k q^{6-3k+k^2} \cdot q^3 p_{12}^k p_{21}^{3-k} q^{(3-k)k} = q^9 \cdot (p_{12} p_{21})^{3-k} (p_{21} p_{12})^k = q^9 q^{-3(3-k)} q^{-3k} = 1.$$

□

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UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, FACULTAD DE ESTUDIOS SUPERIORES CUAUTITLÁN, PRIMERO DE MAYO S/N, CAMPO 1, CIT, CUAUTITLAN IZCALLI, EDSTADO DE MÉXICO, 54768, MEXICO.

FAMAF-CIEM (CONICET), UNIVERSIDAD NACIONAL DE CÓRDOBA, MEDINA ALLEGRE S/N, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, REPÚBLICA ARGENTINA.

*E-mail address:* vlad@unam.mx; vay@famaf.unc.edu.ar